Decidability of Two Truly Concurrent Equivalences for Finite Bounded Petri Nets

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Abstract. We prove that (strong) fully-concurrent bisimilarity and causal-net bisimilarity are decidable for finite bounded Petri nets. The proofs are based on a generalization of the ordered marking proof technique that Vogler used to demonstrate that (strong) fully-concurrent bisimilarity (or, equivalently, history-preserving bisimilarity) is decidable on finite safe nets.

Keywords: Behavioral equivalences · True concurrency · Fully-concurrent bisimilarity · Causal-net bisimilarity · Decidability.

1 Introduction

Truly concurrent equivalences, such as fully-concurrent bisimilarity [2] or causal-net bisimilarity [9,12], have been advocated as very suitable equivalences to compare the behavior of Petri nets. However, most results about their decidability [19,14,20] are limited to the class of finite safe nets, i.e., nets whose places can hold one token at most. Our main aim is to extend some of these decidability proofs to the case of bounded nets.

In his seminal paper [19], Vogler demonstrated that (strong) fully-concurrent bisimilarity is decidable on finite safe nets. His proof is based on an alternative characterization of fully-concurrent bisimulation, called *ordered marking bisimulation* (OM bisimulation, for short), which represents the current global state of the net system as a marking equipped with a pre-ordering on its tokens, reflecting the causal ordering of the transitions that produced the tokens. However, the ordered marking idea works well if the marking is a set (as it is the case for safe nets), and so it is not immediate to generalize it to bounded nets, whose markings are, in general, multisets. The first contribution of this paper is the definition of an alternative token game semantics for Petri nets which is defined according to the *individual token philosophy*, rather than the *collective token philosophy*, as it is customary for Petri nets. This is obtained by representing each token as a pair (s,i), where s is the name of the place where the token is on, and i is a natural number (an index) assigned to the token in such a way that different tokens on the same place have different indexes. In this way, a multiset over the set of places (i.e., a marking) is turned into a set of indexed places. The main advantage of having turned multisets into sets is that Vogler's ordered marking idea can be used also in this richer context, yielding *ordered indexed markings*. The second contribution of the paper is to show that (strong) fully-concurrent bisimulation can be equivalently characterized as a suitable bisimulation over ordered indexed markings, called *OIM bisimulation*, generalizing the approach by Vogler [19]. An OIM bisimulation is formed by a set of triples, each composed of two ordered indexed markings and a relation between these two ordered indexed markings that respects the pre-orders. The decidability of (strong) fully-concurrent bisimilarity on finite bounded nets follows by observing that the reachable indexed markings are finitely many, so the ordered indexed markings of interest are finitely many as well, so that there can only be finitely many candidate relations (which are all finite) to be OIM-bisimulations. The third contribution of the paper is to show that our generalization of Vogler's proof technique can be adapted to prove the decidability also of another truly concurrent behavioral equivalence, namely *causal-net bisimilarity* [12], a behavioral equivalence that coincides with *structure-preserving bisimilarity* [9], and which is slightly finer than (strong) fully-concurrent bisimilarity.

The paper is organized as follows. Section 2 recalls the basic definitions about Petri nets. Section 3 recalls the causal semantics, including the definition of causal-net bisimilarity and (strong) fully-concurrent bisimilarity. Section 4 introduces indexed markings and the alternative, individual, token game semantics. Section 5 describes indexed ordered markings and their properties. Section 6 introduces OIM-bisimulation, proves that its equivalence coincides with (strong) fully-concurrent bisimilarity and, moreover, shows that it is decidable. Section 7 hints that also causal-net bisimilarity is decidable. Finally, Section 8 discusses related literature and some future research. For lack of space, longer proofs are to be found in the preliminary full version of this article [5].

2 Basic Definitions

Definition 1. (Multiset) *Let $\mathbb{N}$ be the set of natural numbers. Given a finite set S, a* multiset *over S is a function $m : S \to \mathbb{N}$. The* support *set $dom(m)$ of m is $\{s \in S \mid m(s) \neq 0\}$. The set of all multisets over S, denoted by $\mathscr{M}(S)$, is ranged over by m. We write $s \in m$ if $m(s) > 0$. The* multiplicity *of s in m is given by the number $m(s)$. The* size *of m, denoted by $|m|$, is the number $\sum_{s \in S} m(s)$, i.e., the total number of its elements. A multiset m such that $dom(m) = \emptyset$ is called* empty *and is denoted by θ. We write $m \subseteq m'$ if $m(s) \leq m'(s)$ for all $s \in S$.* Multiset union *$\_ \oplus \_$ is defined as follows: $(m \oplus m')(s) = m(s) + m'(s)$.* Multiset difference *$\_ \ominus \_$ is defined as follows: $(m_1 \ominus m_2)(s) = max\{m_1(s) - m_2(s), 0\}$. The* scalar product *of a number j with m is the multiset $j \cdot m$ defined as $(j \cdot m)(s) = j \cdot (m(s))$. By s_i we also denote the multiset with s_i as its only element. Hence, a multiset m over $S = \{s_1, \dots, s_n\}$ can be represented as $k_1 \cdot s_1 \oplus k_2 \cdot s_2 \oplus \dots \oplus k_n \cdot s_n$, where $k_j = m(s_j) \geq 0$ for $j = 1, \dots, n$.* □

Definition 2. (Place/Transition net) *A labeled* Place/Transition *Petri net (P/T net for short) is a tuple $N = (S, A, T)$, where*

- *S is the finite set of* places*, ranged over by s (possibly indexed),*
- *A is the finite set of* labels*, ranged over by ℓ (possibly indexed), and*
- *$T \subseteq (\mathscr{M}(S) \setminus \{\theta\}) \times A \times \mathscr{M}(S)$ is the finite set of* transitions*, ranged over by t (possibly indexed).*

Given a transition $t = (m, \ell, m')$, we use the notation:

- $^\bullet t$ *to denote its* pre-set m *(which cannot be an empty multiset) of tokens to be consumed;*
- $l(t)$ *for its* label ℓ*, and*
- $t^\bullet$ *to denote its* post-set m' *of tokens to be produced.*

Hence, transition t can be also represented as $^\bullet t \xrightarrow{l(t)} t^\bullet$. *We also define the* flow function $flow : (S \times T) \cup (T \times S) \to \mathbb{N}$ *as follows: for all* $s \in S$*, for all* $t \in T$, $flow(s,t) = {}^\bullet t(s)$ *and* $flow(t,s) = t^\bullet(s)$. *We will use F to denote the* flow relation $\{(x,y) \in (S \times T) \cup (T \times S) \mid flow(x,y) > 0\}$. *Finally, we define pre-sets and post-sets also for places as follows:* $^\bullet s = \{t \in T \mid s \in t^\bullet\}$ *and* $s^\bullet = \{t \in T \mid s \in {}^\bullet t\}$. □

In the graphical description of finite P/T nets, places (represented as circles) and transitions (represented as boxes) are connected by directed arcs. The arcs may be labeled with the number representing how many tokens of that type are to be removed from (or produced into) that place; no label on the arc is interpreted as the number one, i.e., one token flowing on the arc. This numerical label of the arc is called its *weight*.

Definition 3. (Marking, P/T net system) *A multiset over S is called a* marking*. Given a marking m and a place s, we say that the place s contains* $m(s)$ tokens*, graphically represented by* $m(s)$ *bullets inside place s. A* P/T net system $N(m_0)$ *is a tuple* (S, A, T, m_0)*, where* (S, A, T) *is a P/T net and* m_0 *is a marking over S, called the* initial marking*. We also say that* $N(m_0)$ *is a* marked *net.* □

The sequential semantics of a marked net is defined by the so-called *token game*, describing the flow of tokens through it. There are several possible variants (see, e.g., [8]) of the token game; below we present the so-called *collective interpretation*, according to which multiple tokens on the same place are indistinguishable, while in Section 4 we introduce a novel variant following the so-called *individual interpretation*.

Definition 4. (Token game) *A transition t is* enabled *at m, denoted* $m[t\rangle$*, if* $^\bullet t \subseteq m$*. The* firing *of t enabled at m produces the marking* $m' = (m \ominus {}^\bullet t) \oplus t^\bullet$*, written* $m[t\rangle m'$. □

Definition 5. (Firing sequence, reachable marking) *A* firing sequence *starting at m is defined inductively as follows:*

- $m[\varepsilon\rangle m$ *is a firing sequence (where* ε *denotes an empty sequence of transitions) and*
- *if* $m[\sigma\rangle m'$ *is a firing sequence and* $m'[t\rangle m''$*, then* $m[\sigma t\rangle m''$ *is a firing sequence.*

If $\sigma = t_1 \ldots t_n$ *(for* $n \geq 0$*) and* $m[\sigma\rangle m'$ *is a firing sequence, then there exist* $m_1, \ldots, m_{n+1}$ *such that* $m = m_1[t_1\rangle m_2[t_2\rangle \ldots \ldots m_n[t_n\rangle m_{n+1} = m'$*, and* $\sigma = t_1 \ldots t_n$ *is called a* transition sequence *starting at m and ending at* m'*. The set of* reachable markings *from m is* $[m\rangle = \{m' \mid \exists \sigma. m[\sigma\rangle m'\}$*. Note that the set of reachable markings may be countably infinite for finite P/T nets.* □

Definition 6. (Classes of finite P/T Nets) *A* finite marked *P/T net* $N = (S, A, T, m_0)$ *is:*

- safe *if every place contains at most one token under every reachable marking, i.e.* $\forall s \in S$, $m(s) \leq 1$ *for all* $m \in [m_0\rangle$.
- bounded *if the number of token in any place is bounded by some k for any reachable marking, i.e.* $\exists k \in \mathbb{N}, \forall s \in S$ *such that* $m(s) \leq k$ *for all* $m \in [m_0\rangle$*. If this is the case, we say that the net is k-bounded (hence, a safe net is just a 1-bounded net).* □

3 Causality-based Semantics

We outline some definitions adapted from literature (cf., e.g., [11,1,16,19,9,12]).

Definition 7. (Acyclic net) *A P/T net $N = (S, A, T)$ is* acyclic *if its flow relation F is acyclic (i.e., $\nexists x$ such that $x\, F^+\, x$, where F^+ is the transitive closure of F).*

The concurrent semantics of a marked P/T net is defined by a class of particular acyclic safe nets, where places are not branched (hence they represent a single run) and all arcs have weight 1. This kind of net is called *causal net*. We use the name C (possibly indexed) to denote a causal net, the set B to denote its places (called *conditions*), the set E to denote its transitions (called *events*), and L to denote its labels.

Definition 8. (Causal net) *A causal net is a finite marked net $C(\mathsf{m}_0) = (B, L, E, \mathsf{m}_0)$ satisfying the following conditions:*

1. *C is acyclic;*
2. *$\forall b \in B\ \ |{}^\bullet b| \leq 1 \wedge |b^\bullet| \leq 1$ (i.e., the places are not branched);*
3. *$\forall b \in B\ \ \mathsf{m}_0(b) = \begin{cases} 1 & \text{if } {}^\bullet b = \emptyset \\ 0 & \text{otherwise;} \end{cases}$*
4. *$\forall e \in E\ \ {}^\bullet e(b) \leq 1 \wedge e^\bullet(b) \leq 1$ for all $b \in B$ (i.e., all the arcs have weight 1).*

We denote by $Min(C)$ the set m_0, and by $Max(C)$ the set $\{b \in B \mid b^\bullet = \emptyset\}$.
A sequence of events $\sigma \in E^*$ is maximal *(or* complete*) if it contains all events in E, each taken once only.* □

Note that any reachable marking of a causal net is a set, i.e., this net is *safe*. As the initial marking of a causal net is fixed by its shape, in the following, in order to make the notation lighter, we often omit the indication of the initial marking (also in their graphical representation), so that the causal net $C(\mathsf{m}_0)$ is denoted by C.

Definition 9. (Moves of a causal net) *Given two causal nets $C = (B, L, E, \mathsf{m}_0)$ and $C' = (B', L, E', \mathsf{m}_0)$, we say that C moves in one step to C' through e, denoted by $C[e\rangle C'$, if ${}^\bullet e \subseteq Max(C)$, $E' = E \cup \{e\}$ and $B' = B \cup e^\bullet$.* □

Definition 10. (Folding and Process) *A* folding *from a causal net $C = (B, L, E, \mathsf{m}_0)$ into a net system $N(m_0) = (S, A, T, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:*

- *$L = A$ and $\mathsf{l}(e) = l(\rho(e))$ for all $e \in E$;*
- *$\rho(\mathsf{m}_0) = m_0$, i.e., $m_0(s) = |\rho^{-1}(s) \cap \mathsf{m}_0|$;*
- *$\forall e \in E, \rho({}^\bullet e) = {}^\bullet\rho(e)$, i.e., $\rho({}^\bullet e)(s) = |\rho^{-1}(s) \cap {}^\bullet e|$ for all $s \in S$;*
- *$\forall e \in E, \rho(e^\bullet) = \rho(e)^\bullet$, i.e., $\rho(e^\bullet)(s) = |\rho^{-1}(s) \cap e^\bullet|$ for all $s \in S$.*

A pair (C, ρ), where C is a causal net and ρ a folding from C to a net system $N(m_0)$, is a process *of $N(m_0)$, written also as π.* □

Definition 11. (Partial orders of events from a process) *From a causal net $C = (B, L, E, \mathsf{m}_0)$, we can extract the* partial order of its events $\mathsf{E}_\mathsf{C} = (E, \preceq)$*, where $e_1 \preceq e_2$ if there is a path in the net from e_1 to e_2, i.e., if $e_1 \mathsf{F}^* e_2$, where F^* is the reflexive and transitive closure of F, which is the flow relation for C. Given a process $\pi = (C, \rho)$, we denote $\preceq$ as $\leq_\pi$, i.e. given $e_1, e_2 \in E$, $e_1 \leq_\pi e_2$ if and only if $e_1 \preceq e_2$.* □

Definition 12. (Moves of a process) *Let $N(m_0) = (S, A, T, m_0)$ be a net system and let (C_i, ρ_i), for $i = 1, 2$, be two processes of $N(m_0)$. We say that (C_1, ρ_1) moves in one step to (C_2, ρ_2) through e, denoted by $(C_1, \rho_1) \xrightarrow{e} (C_2, \rho_2)$, if $C_1[e\rangle C_2$ and $\rho_1 \subseteq \rho_2$.*
If $\pi_1 = (C_1, \rho_1)$ and $\pi_2 = (C_2, \rho_2)$, we denote the move as $\pi_1 \xrightarrow{e} \pi_2$. □

Definition 13. (Causal-net bisimulation) *Let $N = (S, A, T)$ be a finite P/T net. A* causal-net bisimulation *is a relation R, composed of triples of the form (ρ_1, C, ρ_2), where, for $i = 1, 2$, (C, ρ_i) is a process of $N(m_{0_i})$ for some m_{0_i}, such that if $(\rho_1, C, \rho_2) \in R$ then*

i) $\forall t_1, C', \rho'_1$ *such that* $(C, \rho_1) \xrightarrow{e} (C', \rho'_1)$, *where* $\rho'_1(e) = t_1$, $\exists t_2, \rho'_2$ *such that* $(C, \rho_2) \xrightarrow{e} (C', \rho'_2)$, *where* $\rho'_2(e) = t_2$, *and* $(\rho'_1, C', \rho'_2) \in R$;
ii) *symmetrically,* $\forall t_2, C', \rho'_2$ *such that* $(C, \rho_2) \xrightarrow{e} (C', \rho'_2)$, *where* $\rho'_2(e) = t_2$, $\exists t_1, \rho'_1$ *such that* $(C, \rho_1) \xrightarrow{e} (C', \rho'_1)$, *where* $\rho'_1(e) = t_1$, *and* $(\rho'_1, C', \rho'_2) \in R$.

Two markings m_1 and m_2 of N are cn-bisimilar (or cn-bisimulation equivalent), denoted by $m_1 \sim_{cn} m_2$, if there exists a causal-net bisimulation R containing a triple $(\rho^0_1, C^0, \rho^0_2)$, where C^0 contains no events and $\rho^0_i(Min(C^0)) = \rho^0_i(Max(C^0)) = m_i$ for $i = 1, 2$. □

Causal-net bisimilarity, which coincides with *structure-preserving bisimilarity* [9], observes not only the events, but also the structure of the distributed state. A weaker equivalence, observing only the events performed, is *fully-concurrent bisimulation* (fc-bisimulation, for short) [2] , whose definition was inspired by previous notions of equivalence on other models of concurrency [17,6,10].

Definition 14. (Fully-concurrent bisimilarity) *Given a finite P/T net $N = (S, A, T)$, a* fully-concurrent bisimulation *is a relation R, composed of triples of the form (π_1, f, π_2) where, for $i = 1, 2$, $\pi_i = (C_i, \rho_i)$ is a process of $N(m_{0i})$ for some m_{0i} and f is an isomorphism between E_{C_1} and E_{C_2}, such that if $(\pi_1, f, \pi_2) \in R$ then:*

i) $\forall t_1, \pi'_1$ *such that* $\pi_1 \xrightarrow{e_1} \pi'_1$, *where* $\rho'_1(e_1) = t_1$, *there exist* e_2, t_2, π'_2, f' *such that*
  1. $\pi_2 \xrightarrow{e_2} \pi'_2$ *where* $\rho'_2(e_2) = t_2$,
  2. $f' = f \cup \{e_1 \mapsto e_2\}$,
  3. $(\pi'_1, f', \pi'_2) \in R$;

ii) *symmetrically, if π_2 moves first.*

Two markings m_1, m_2 of N are fc-bisimilar, denoted by $m_1 \sim_{fc} m_2$ if a fully-concurrent bisimulation R exists, containing a triple $(\pi^0_1, \emptyset, \pi^0_2)$ where $\pi^0_i = (C^0_i, \rho^0_i)$ such that C^0_i contains no events and $\rho^0_i(Min(C^0_i)) = \rho^0_i(Max(C^0_i)) = m_i$ for $i = 1, 2$. □

4 Indexed Marking Semantics

We define an alternative, novel token game semantics for Petri nets according to the *individual token philosophy*. A token is represented as an *indexed place*, i.e. a pair (s,i), where s is the name of the place where the token is on, and i is an index assigned to the token such that different tokens on the same place have different indexes. In this way, a standard marking is turned into an *indexed* marking, i.e. a set of indexed places.

Definition 15. (Indexed marking) *Given a finite net $N = (S,A,T)$, an* indexed marking *is a function $k : S \to \mathscr{P}_{fin}(\mathbb{N})$ associating to each place a finite set of natural numbers, such that the associated (de-indexed) marking m is obtained as $m(s) = |k(s)|$ for each $s \in S$. In this case, we write $\alpha(k) = m$. The support set $dom(k)$ is $\{s \in S \mid k(s) \neq \emptyset\}$. The set of the indexed markings with support set S is denoted by $\mathfrak{K}(S)$.*

An indexed place is a pair (s,i) such that $s \in S$ and $i \in \mathbb{N}$. A set of indexed places $\{(s_1,i_1),\ldots,(s_n,i_n)\} \in \mathscr{P}(S \times \mathbb{N})$ is also another way of describing an indexed marking.[1] *Hence, $\mathfrak{K}(S) \subseteq \mathscr{P}(S \times \mathbb{N})$. Each element of an indexed marking, i.e. each indexed place, is a token.*

An indexed marking $k \in \mathfrak{K}(S)$ is closed *if $k(s) = \{1,\ldots,|k(s)|\}$ for all $s \in dom(k)$. If there exists a marked net $N(m_0)$ and a closed indexed marking k_0 such that $\alpha(k_0) = m_0$, we say that k_0 is an* initial indexed *marking of N, and we write $N(k_0)$.* □

We define the difference between an indexed marking k and a marking m (such that $m(s) \leq |k(s)|$ for all $s \in S$) as $\boxminus : \mathfrak{K}(S) \to \mathscr{M}(S) \to \mathscr{P}(\mathfrak{K}(S))$

$$\begin{aligned} k \boxminus \theta &= \{k\} \\ k \boxminus (s \oplus m) &= (k \boxminus s) \boxminus m \\ \{k_1, \ldots k_n\} \boxminus m &= k_1 \boxminus m \cup \ldots \cup k_n \boxminus m \\ k \boxminus s = \{k' \mid k'(s') &= k(s') \text{ if } s' \neq s \text{ ,or} \\ &= k(s) \setminus \{n\} \text{ if } s' = s \text{ and } n \in k(s)\} \end{aligned}$$

and the union of an indexed marking k and a marking m as $\boxplus : \mathfrak{K}(S) \to \mathscr{M}(S) \to \mathfrak{K}(S)$

$$\begin{aligned} k \boxplus \theta &= k \\ k \boxplus (s \oplus m) &= (k \boxplus s) \boxplus m \\ k \boxplus s &= k' \end{aligned}$$

where for all $s' \in S$, $k'(s')$ is defined as:

$$k'(s') = \begin{cases} k(s') & \text{if } s' \neq s \\ k(s) \cup \{n\} & \text{if } s' = s,\ n = min(\overline{k(s)}) \text{ where } \overline{k(s)} = \mathbb{N} \setminus k(s) \end{cases}$$

where we use $min(H)$, with $H \in \mathscr{P}(\mathbb{N})$, to denote the least element of H. Note that the difference between an indexed marking and a marking is a *set* of indexed markings:

[1] Being a set, we are sure that $\nexists j_1, j_2$ such that $s_{j_1} = s_{j_2} \wedge i_{j_1} = i_{j_2}$, i.e., each token on a place s has an index different from the index of any other token on s.

since it makes no sense to prefer a single possible execution over another, all possible choices for $n \in k(s)$ are to be considered. The token game is modified accordingly, taking into account the individual token interpretation.

Definition 16. (Token game with indexed markings) *Given a net $N = (S, A, T)$ and an indexed marking $k \in \mathfrak{K}(S)$ such that $m = \alpha(k)$, we say that a transition $t \in T$ is* enabled *at k if $^{\bullet}t \subseteq m$, denoted $k[\![t\rangle$. If t occurs, the firing of t enabled at k produces the indexed marking k', denoted $k[\![t\rangle k'$, if*

- *$\exists k'' \in k \boxminus {}^{\bullet}t$ and*
- *$k' = k'' \boxplus t^{\bullet}$.* □

Note that there can be more than one indexed marking produced by the firing of t, but for all k' such that $k[\![t\rangle k'$, it is true that $\alpha(k') = m \ominus {}^{\bullet}t \oplus t^{\bullet}$.

From now on, indexed markings will be always represented as sets of indexed places, i.e., we denote an indexed marking k by $\{(s_1, n_1) \dots (s_i, n_i)\}$ where $|k| = i$.

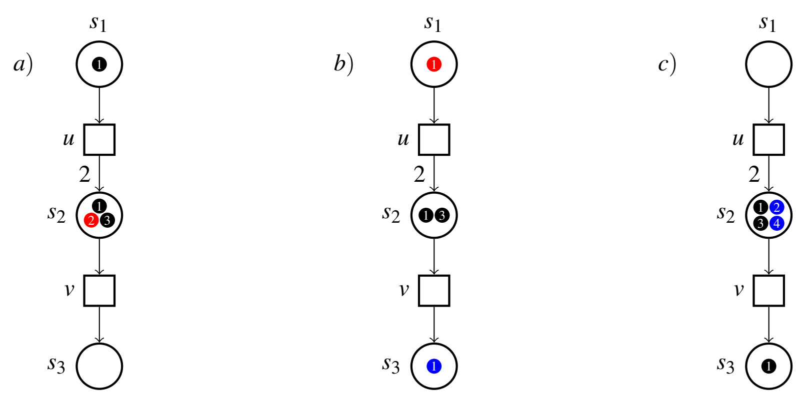

Fig. 1. Execution of the transition labeled by v, then of the transition labeled by u, on a bounded net with initial marking $m_0 = s_1 \oplus 3s_2$. Tokens to be consumed are in red, generated ones in blue.

Example 1. In Figure 1(a) a simple net N is given. The initial marking is $m_0 = s_1 \oplus 3s_2$, therefore the marked net $N(m_0)$ is 5-bounded. The initial indexed marking is $k_0 = \{(s_1,1),(s_2,1),(s_2,2),(s_2,3)\}$. Let us suppose that transition t_2, labeled by v, occurs. There are three possible ways to remove a token from s_2: removing $(s_2,1)$, or removing $(s_2,2)$, or removing $(s_2,3)$. Indeed, the operation $k_0 \boxminus {}^{\bullet}t_2$ yields a set of three possible indexed markings, each one a possible result of the difference: $\{\{(s_1,1),(s_2,2),(s_2,3)\}$, $\{(s_1,1),(s_2,1),(s_2,3)\}, \{(s_1,1),(s_2,1),(s_2,2)\}\}$. Let us choose, for the sake of the argument, that the token deleted by t_2 is $(s_2,2)$, i.e. choose $k' = \{(s_1,1),(s_2,1),(s_2,3)\}$. The union $k' \boxplus t_2^{\bullet}$ easily yields the indexed marking $k_1 = \{(s_1,1),(s_2,1),(s_2,3),(s_3,1)\}$, as depicted in Figure 1(b). Note that this choice was arbitrary and two other values of k_1 are possible. Indeed, from Definition 16, we know that the transition relation

	generated	deleted	untouched
$m[t\rangle m'$	$t^\bullet$	$^\bullet t$	$m \ominus {}^\bullet t$
$k[\![t\rangle k'$	$k' \setminus k''$	$k \setminus k''$	k''

Table 1. Different notation for tokens in the token game. On the first line, the collective case. On the last one, the individual case.

on indexed markings is nondeterministic. However, the resulting marked net is the same for all three cases, that is, the same of Figure 1(b) without indexes. Now we suppose that (given the indexed marking k_1 from above) transition t_1, labeled by u, occurs. In that case, $k_1 \boxminus {}^\bullet t_1$ yields the singleton set $\{\{(s_2,1),(s_2,3),(s_3,1)\}\}$, therefore we choose $k'' = \{(s_2,1),(s_2,3),(s_3,1)\}$. Since $t_1^\bullet = s_2 \oplus s_2$, we show in detail how $k'' \boxplus t_1^\bullet$ is computed. First, we apply the definition for union with non-singleton multisets: $k'' \boxplus (s_2 \oplus s_2) = (k'' \boxplus s_2) \boxplus s_2$. Then, we compute $k'' \boxplus s_2$: since the least free index for the place s_2 is 2, $k'' \boxplus s_2 = \{(s_2,1),(s_2,2)(s_2,3),(s_3,1)\}$. Now we apply again the definition: note that this time the least free index for s_2 is 4, and the final result is $k_2 = \{(s_2,1),(s_2,2)(s_2,3),(s_2,4),(s_3,1)\}$. The resulting marked net is depicted in Figure 1(c). □

The notation for tokens in the token game has become slightly more unintuitive, so in Table 1 we provide a comparison between the one used in the previous sections and the one we will use in the following part of this work. Given a transition t such that $k[\![t\rangle k'$ and $m[t\rangle m'$, assume $k'' \in k \boxminus {}^\bullet t$ such that $k' = k'' \boxplus t^\bullet$, where $\alpha(k) = m$ and $\alpha(k') = m'$.

Definition 17. (Firing sequence with IM) *Given a finite net $N = (S,A,T)$ and an indexed marking k, a* firing sequence *starting at k is defined inductively as follows:*

- *$k[\![\varepsilon\rangle k$ is a firing sequence (where ε denotes an empty sequence of transitions) and*
- *if $k[\![\sigma\rangle k'$ is a firing sequence and $k'[\![t\rangle k''$, then $k[\![\sigma t\rangle k''$ is a firing sequence.*

The set of reachable indexed markings from k is $[\![k\rangle = \{k' \mid \exists\sigma\,.\,k[\![\sigma\rangle k'\}$. Given a net $N(k_0)$, we call $[\![k_0\rangle$ the set of reachable indexed marking of N, denoted by $IM(N)$. □

Proposition 1. *Given a finite bounded net $N = (S,A,T,m_0)$, the set $IM(N) \subseteq \mathfrak{K}(S)$ of reachable indexed markings is finite.*

Proof. Full detail in the preliminary full version of this article [5]. □

5 Ordered Indexed Marking Semantics

Vogler [19] introduces *ordered markings* (OM for short) to describe the state of a safe marked net. They consist of a safe marking together with a preorder which reflects precedences in the generation of tokens. This is reflected in the token game for OM: if s precedes some s'' in the old OM and s'' is used to produce a new token s', then s must precede s' in the new OM.

The key idea of Vogler's decidability proof for safe nets is that the OM obtained by a sequence of transitions of a net is the same as the one induced by a process, whose events correspond to that sequence of transitions, on the net itself. Since ordered markings are finite objects, Vogler defines OM-bisimulation and shows that it coincides with fully-concurrent bisimulation. He himself hinted at a possibility [19] of extending the result to bounded nets, but suggested that it would have been technically quite involved.

We adapt his approach by defining a semantics based on *ordered indexed markings*, taking into account the individual token interpretation of nets, and proving that an extension to bounded nets is indeed possible.

Definition 18. **(Ordered indexed marking)** *Given a P/T net $N = (S,A,T)$ and an indexed marking $k \in \mathfrak{K}(S)$, the pair $(k,\leq)$ is an ordered indexed marking if $\leq \subseteq k \times k$ is a preorder, i.e. a reflexive and transitive relation. The set of all possible ordered indexed markings of N is denoted by $OIM(N)$.*

If k_0 is the initial indexed marking of N, we define the initial ordered indexed marking *$init(N)$ as $(k_0, k_0 \times k_0)$. If the initial indexed marking is not clear from the context, we write $init(N(k_0))$ to denote the initial ordered indexed marking.* □

Definition 19. **(Token game with ordered indexed markings)** *Given a P/T net $N = (S,A,T)$ and an ordered indexed marking $(k,\leq)$, we say that a transition $t \in T$ is* enabled *at $(k,\leq)$ if $k[\![t\rangle$; this is denoted by $(k,\leq)[\![t\rangle$.*

The firing of t enabled at $(k,\leq)$ may produce an ordered indexed marking $(k',\leq')$ – and we denote this by $(k,\leq)[\![t\rangle(k',\leq')$ – where:

- *$k'' \in k \boxminus {}^\bullet t$ such that $k' = k'' \boxplus t^\bullet$*
- *for all $(s_h,i_h),(s_j,i_j) \in k'$, $(s_h,i_h) \leq' (s_j,i_j)$ if and only if:*
  1. *$(s_h,i_h),(s_j,i_j) \in k''$ and $(s_h,i_h) \leq (s_j,i_j)$, or*
  2. *$(s_h,i_h),(s_j,i_j) \in k' \setminus k''$, or*
  3. *$(s_h,i_h) \in k''$, $(s_j,i_j) \in k' \setminus k''$ and $\exists (s_l,i_l) \in k \setminus k''$ such that $(s_h,i_h) \leq (s_l,i_l)$.* □

Note that, as for indexed markings, many different ordered indexed markings are produced from the firing of t. This means that also the transition relation for ordered indexed markings is nondeterministic. Moreover, in the same fashion as Vogler's work [19], the preorder reflects the precedence in the generation of tokens, which is not strict, i.e. if tokens (s_1,n_1) and (s_2,n_2) are generated together we have both $(s_1,n_1) \leq (s_2,n_2)$ and $(s_2,n_2) \leq (s_1,n_1)$.

Example 2. Consider again the net in Figure 1 and the first part of the execution of Example 1, i.e. $k_0[\![t_2\rangle k_1$. According to Definition 18, the initial ordered indexed marking is $(k_0,\leq_0)$, where $\leq_0 = k_0 \times k_0$. When t_2 fires, token $(s_2,2)$ is removed and token $(s_3,1)$ is generated, while all other tokens are untouched. Let us denote the preorder induced by the firing of t_2 as $\leq_1$. According to item 2 of Definition 19, since $(s_3,1)$ is generated by the firing of t_2, we have $(s_3,1) \leq_1 (s_3,1)$. According to item 1 of Definition 19, the preorder on all tokens untouched by t_2 remains the same, therefore e.g. $(s_2,3) \leq_1 (s_1,1)$ and viceversa. Furthermore, consider $(s_1,1)$ and $(s_3,1)$: we have that t_2 generates $(s_3,1)$, deletes $(s_2,2)$ and leaves $(s_1,1)$ untouched. Since $(s_1,1) \leq_0 (s_2,2)$, by item 3 of Definition 19 we have $(s_1,1) \leq_1 (s_3,1)$. The same reasoning applies to all untouched

tokens. Summing up, we have $(k_0, \leq_0)[\![t_2\rangle(k_1, \leq_1)$ where
$\leq_1 = \leq_0 \setminus \{((s_i,n_i),(s_j,n_j)) \in k_0 \mid (s_i,n_i) = (s_2,2) \vee (s_j,n_j) = (s_2,2)\} \cup$
$\{((s_1,1),(s_3,1)),((s_2,1),(s_3,1)),((s_2,3),(s_3,1)),((s_3,1),(s_3,1))\}$. □

Definition 20. (Firing sequence with OIM) *A* firing sequence *starting at* $(k, \leq)$ *is defined inductively as follows:*

- $(k, \leq)[\![\varepsilon\rangle(k, \leq)$ *is a firing sequence (where* ε *denotes an empty sequence of transitions) and*
- *if* $(k, \leq)[\![\sigma\rangle(k', \leq')$ *is a firing sequence and* $(k', \leq')'[\![t\rangle(k'', \leq'')$*, then* $(k, \leq)[\![\sigma t\rangle(k'', \leq'')$ *is a firing sequence.*

The set of reachable ordered indexed markings from $(k, \leq)$ *is*
$[\![(k, \leq)\rangle = \{(k', \leq') \mid \exists \sigma \,.\, (k, \leq)[\![\sigma\rangle(k', \leq')\}$.
Given an initial indexed marking k_0*, the set of all the reachable ordered indexed markings of* $N(k_0)$ *is denoted by* $[\![init(N)\rangle$. □

Proposition 2. *Given a bounded net* $N = (S, A, T, m_0)$*,* $[\![init(N)\rangle$ *is finite.*

Proof. The set $IM(N)$ of reachable indexed markings is finite by Proposition 1. The set of possible preorders for an indexed marking $k = \{(s_1,n_1)\dots(s_j,n_j)\} \in IM(N)$ is finite, because $\leq \subseteq k \times k$. Therefore, $[\![init(N)\rangle$ is finite. □

5.1 Ordered indexed marking and causality-based semantics

Given a transition sequence σ, there is an operational preorder on tokens obtained by Definition 20, and a preorder derived from the process $\leq_\pi$ obtained from the causal net C corresponding to the transition sequence σ. In the following, we introduce some notation for processes and ordered indexed markings, and relate the two. If $\pi = (C, \rho)$ is a process of a marked net $N(m_0)$ and k_0 is the initial indexed marking for $N(m_0)$ (i.e. $\alpha(k_0) = m_0$ and k_0 is closed), we also say that π is a process of $N(k_0)$. Moreover, given a process $\pi = (C, \rho)$ of a net $N(k_0)$ and σ a complete transition sequence of C, we write $init(N)[\![\pi\rangle(k, \leq)$ if $init(N)[\![\rho(\sigma)\rangle(k, \leq)$.

Theorem 1. *Let* $\pi = (C, \rho)$ *a process of* $N(k_0)$ *such that* $init(N)[\![\pi\rangle(k, \leq)$.
Then $(k, \leq)[\![t\rangle(k', \leq')$ *if and only if* $\pi \xrightarrow{e} \pi'$ *where* $\rho'(e) = t$ *and* $init(N)[\![\pi'\rangle(k', \leq')$.

Proof. By induction on the sequence $\sigma.e$ where σ is a complete transition sequence of C. Complete proof in the preliminary full version of this article [5]. □

Example 3. In Figure 2(a), the same 5-bounded P/T net N as Figure 1 is depicted, together with its empty process (we omit the representation of its initial marking). Figure 2(b,c) shows how the process corresponding to the transition sequence $t_2 t_1$ grows. We consider the same execution as in Example 1, i.e. $k_0[\![t_2\rangle k_1[\![t_1\rangle k_2$. For simplicity's sake, in the following each condition will be mapped to the place having same subscript and each event will be mapped to the transition having same label. We will denote each process π_i as the one thus corresponding to causal net C_i. Before any transition fires, we have $init(N) = (k_0, \leq_0)$ where $\leq_0 = k_0 \times k_0$ by Definition 18. Not surprisingly, all

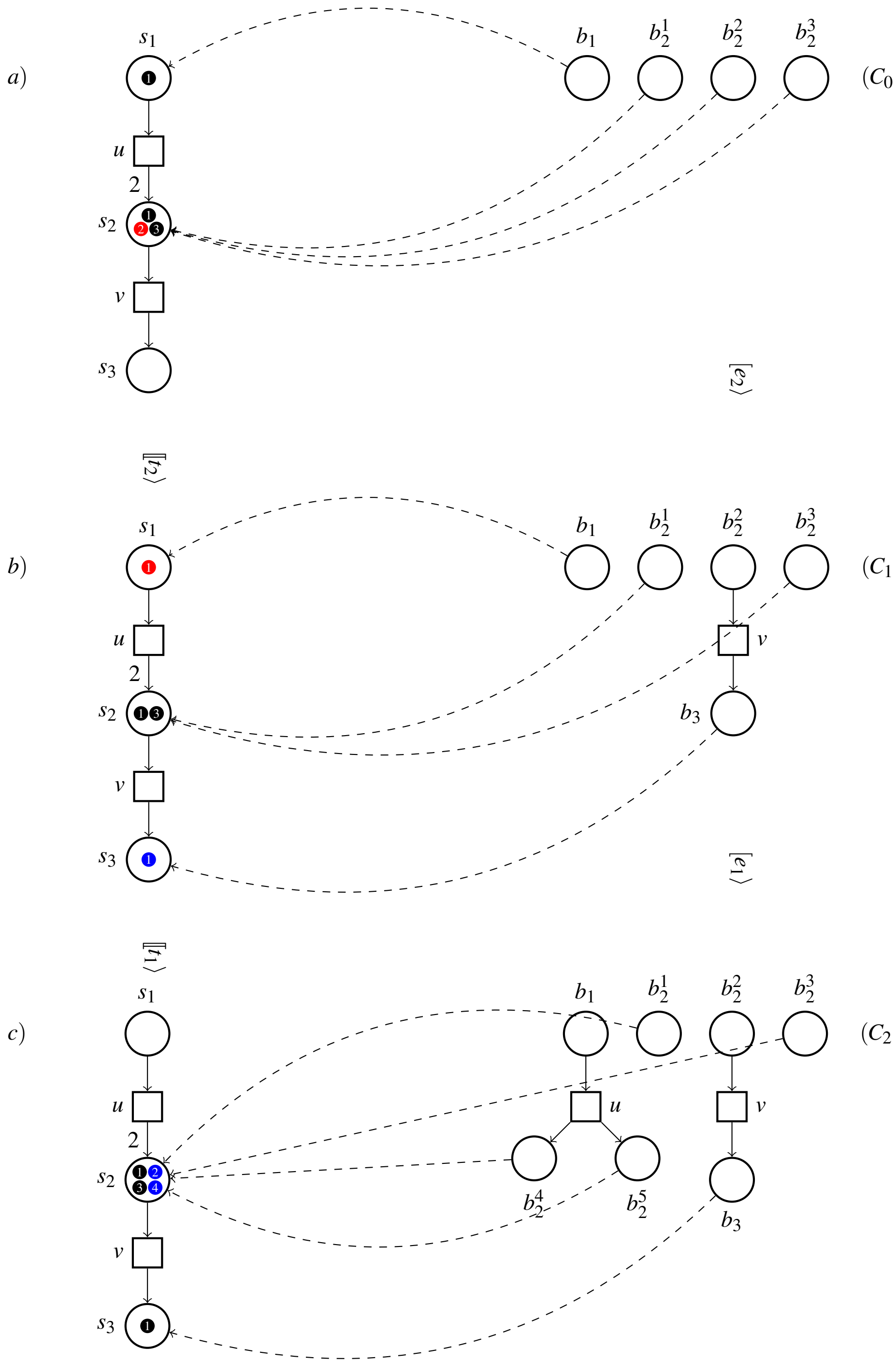

Fig. 2. Execution of the transition labeled by v, then u, on the net of Figure 1 and corresponding process (only the mapping of maximal conditions to tokens is displayed). Tokens to be consumed are red, generated ones blue.

conditions b_i^j are minimal in the causal net C_0 and mapped to tokens in the initial indexed ordered marking. The firing of t_2 deletes token $(s_2,2)$ and generates token $(s_3,1)$; moreover, since $(s_2,1) \leq_0 (s_2,2)$ we have $(s_2,1) \leq_1 (s_3,1)$. Note that $b_2^1 \in Min(C_1)$ but $b_3 \notin Min(C_1)$. After the firing of t_1, there are four tokens in place s_2. However, since $(s_2,2)$ and $(s_2,4)$ are generated by t_1, they are greater in $\leq_2$ than $(s_2,1)$ and $(s_2,3)$. This can also be seen at the process level: b_2^1 and b_2^3 are minimal conditions of C_2, while b_2^4 and b_2^5 are not. On the other hand, note that, just as b_2^4 and b_3 are not minimal in C_2 but also not related by $\leq_{\pi_2}$, also $(s_2,2)$ and $(s_3,1)$ are not related by $\leq_2$. □

6 Fully-concurrent Bisimilarity is Decidable

We now define a novel bisimulation relation based on ordered indexed markings, generalizing the similar idea in [19].

Definition 21. (OIM-bisimulation) *Let $N = (S,A,T)$ be a P/T net. An OIM-bisimulation is a relation $\mathfrak{B} \subseteq OIM(N) \times OIM(N) \times \mathscr{P}((S \times \mathbb{N}) \times (S \times \mathbb{N}))$ such that if $((k_1,\leq_1),(k_2,\leq_2),\beta) \in \mathfrak{B}$, then:*

- *$\forall t_1, k_1', \leq_1'$ such that $(k_1,\leq_1)[\![t_1\rangle(k_1',\leq_1')$, (where we assume $k_1'' \in k_1 \boxminus {}^\bullet t_1$ such that $k_1' = k_1'' \boxplus t_1^\bullet$), there exist $t_2', k_2', \leq_2'$ (where we assume $k_2'' \in k_2 \boxminus {}^\bullet t_2$ such that $k_2' = k_2'' \boxplus t_2^\bullet$), and for β' defined as $\forall (s_1,n_1) \in k_1', \forall (s_2,n_2) \in k_2'$:*

$$(s_1,n_1)\ \beta'\ (s_2,n_2) \Leftrightarrow \begin{cases} (s_1,n_1) \in k_1'', (s_2,n_2) \in k_2'', (s_1,n_1)\ \beta\ (s_2,n_2), \text{ or} \\ (s_1,n_1) \in k_1' \setminus k_1'', (s_2,n_2) \in k_2' \setminus k_2'' \end{cases}$$

  the following hold:
  - *$(k_2,\leq_2)[\![t_2\rangle(k_2',\leq_2')$ where $((k_1',\leq_1'),(k_2',\leq_2'),\beta') \in \mathfrak{B}$ and $l(t_1) = l(t_2)$;*
  - *$\forall (s_1,n_1) \in k_1 \setminus k_1''$, $\exists (s_1',n_1') \in k_1 \setminus k_1''$, $(s_2',n_2') \in k_2 \setminus k_2''$ such that $(s_1,n_1) \leq_1 (s_1',n_1') \wedge (s_1',n_1')\ \beta\ (s_2',n_2')$ and symmetrically $\forall (s_2,n_2) \in k_2 \setminus k_2''$, $\exists (s_2',n_2') \in k_2 \setminus k_2''$, $(s_1',n_1') \in k_1 \setminus k_1''$ such that $(s_2,n_2) \leq_2 (s_2',n_2') \wedge (s_1',n_1')\ \beta\ (s_2',n_2')$*
- *symmetrically, if $(k_2,\leq_2)$ moves first.*

Two markings m_1 and m_2 of N are OIM-bisimilar, denoted $m_1 \sim_{oim} m_2$, if there exists an OIM-bisimulation $\mathfrak{B}$ containing the triple $(init(N(k_1)), init(N(k_2), k_1 \times k_2)$ where, for $i = 1,2$, k_i is the initial indexed marking such that $m_i = \alpha(k_i)$. □

Our aim is to relate $\sim_{fc}$ and $\sim_{oim}$ and the idea is that two tokens are related by β if and only if the transition generating one of the two is mapped by f to the transition generating the other one. Next, we show that fully-concurrent bisimilarity and OIM-bisimilarity coincide on P/T nets.

Theorem 2. (OIM-bisimilarity and FC-bisimilarity coincide) *Let $N = (S,A,T)$ be a P/T net and m_1, m_2 two markings of N. $m_1 \sim_{oim} m_2$ if and only if $m_1 \sim_{fc} m_2$.*

Proof. See the preliminary full version of this article [5]. □

Theorem 3. (FC-bisimilarity is decidable for finite bounded nets) *Given $N(m_1)$ and $N(m_2)$ bounded nets, it is decidable to check whether $m_1 \sim_{fc} m_2$.*

Proof. By Theorem 2, we have to check whether $m_1 \sim_{oim} m_2$. There are finitely many possible sets of triples following Definition 21. Therefore we can check by exhaustive search whether one of them is an OIM-bisimulation. Full detail in the preliminary full version of this article [5]. □

We conclude this section with a remark on the complexity of the decision procedure. Assume that the considered net has (less than) s places, t transitions and it is h-bounded. The upper bound for our decision procedure is $2^{O(hs \cdot log(hs)+log(t))}$. Full detail in the preliminary full version of this article [5]. Note that our exhaustion algorithm has no worse complexity than other proposed algorithms [15,14].

7 Causal-net Bisimilarity is Decidable

In the same fashion as the preceding section, we now prove that also causal-net bisimilarity is decidable.

Definition 22. (OIMC bisimulation) *Let $N = (S,A,T)$ be a P/T net. An OIMC bisimulation is a relation $\mathfrak{B} \subseteq OIM(N) \times OIM(N) \times \mathscr{P}((S \times \mathbb{N}) \times (S \times \mathbb{N}))$ such that if $((k_1, \leq_1),(k_2, \leq_2), \beta) \in \mathfrak{B}$, then:*

- $|k_1| = |k_2|$
- $\forall t_1, k_1', \leq_1'$ *if* $(k_1, \leq_1)[\![t_1\rangle(k_1', \leq_1')$ *(where we assume that* $k_1'' \in k_1 \boxminus {}^\bullet t_1$ *such that* $k_1' = k_1'' \boxplus t_1^\bullet$*), then there exist* $t_2, k_2', \leq_2'$ *(where we assume* $k_2'' \in k_2 \boxminus {}^\bullet t_2$ *such that* $k_2' = k_2'' \boxplus t_2^\bullet$*), and for* β' *defined as* $\forall (s_1,n_1) \in k_1', \forall (s_2,n_2) \in k_2'$*:*

$$(s_1,n_1)\ \beta'\ (s_2,n_2) \Leftrightarrow \begin{cases} (s_1,n_1) \in k_1'', (s_2,n_2) \in k_2'', (s_1,n_1)\ \beta\ (s_2,n_2), \textit{or} \\ (s_1,n_1) \in k_1' \setminus k_1'', (s_2,n_2) \in k_2' \setminus k_2'' \end{cases}$$

  the following hold:
  - $(k_1 \setminus k_1'')$ *and* $(k_2 \setminus k_2'')$ *are bijectively related by* β,
  - $(k_2, \leq_2)[\![t_2\rangle(k_2', \leq_2')$ *where* $((k_1', \leq_1'),(k_2', \leq_2'), \beta') \in \mathfrak{B}$ *and* $l(t_1) = l(t_2)$.
- *symmetrically, if* $(k_2, \leq_2)$ *moves first.*

Two markings m_1 and m_2 of N are OIMC bisimilar, denoted $m_1 \sim_{oimc} m_2$, if there exists an OIMC bisimulation $\mathfrak{B}$ containing the triple $(init(N(k_1)), init(N(k_2), k_1 \times k_2)$ where, for $i = 1,2$, k_i is the closed indexed marking such that $m_i = \alpha(k_i)$. □

Theorem 4. (OIMC-bisimilarity and CN-bisimilarity coincide) *Let $N = (S,A,T)$ be a P/T net and m_1, m_2 two markings of N. $m_1 \sim_{oimc} m_2$ if and only if $m_1 \sim_{cn} m_2$.*

Proof. See the preliminary full version of this article [5]. □

Theorem 5. (CN-bisimilarity is decidable for finite bounded nets) *Given $N(m_1)$ and $N(m_2)$ bounded nets, it is decidable to check whether $m_1 \sim_{cn} m_2$.*

Proof. By Theorem 4 we have to check whether there exists an OIMC-bisimulation $\mathfrak{B}$ for the given net N and initial markings m_1, m_2. The proof then follows the same steps of Theorem 3. □

Note that the complexity of this procedure is again $2^{O(hs \cdot log(hs) + log(t))}$.

8 Conclusion and Future Research

We have extended Vogler's proof technique in [19], based on ordered markings, that he used to prove decidability of (strong) fully-concurrent bisimilarity for safe nets, to bounded nets by means of indexed ordered markings. The extension is flexible enough to be applicable also to other similar equivalences, such as causal-net bisimilarity [9,12]. While decidability of fully-concurrent bisimilarity for bounded nets was already proved by Montanari and Pistore [15], our result for causal-net bisimilarity is new. However, the approach of [15] is not defined directly on Petri nets, rather it exploits an encoding of Petri nets into so-called *causal automata*, a model of computation designed for handling dependencies between transitions by means of names. In addition to this, their encoding works modulo isomorphisms, so that, in order to handle correctly the dependency names, at each step of the construction costly renormalizations are required. Along the same line, recently *history-dependent automata* [4,3] have been proposed. They are a much refined version of causal automata, retaining not only events but also their causal relations. Moreover, they are equipped with interesting categorical properties such as having symmetry groups over them, which allow for state reductions. As in the former work, the latter ones do not work directly on the net and may require minimizations (albeit *automatic*, in the case of HD automata). On the contrary, our construction is very concrete and works directly on the net. Thus, we conjecture that, even if the worst-case complexity is roughly the same, our algorithm performs generally better.

Decidability of fully-concurrent bisimilarity for bounded nets using the ordered indexed marking idea was claimed to have been proved by Valero-Ruiz in his PhD thesis [18]. However, his proof contains many flaws; an accurate analysis of which can be found in the preliminary version of this paper [5]. Therefore our work can be considered the first one to have proved it using the ordered indexed marking approach.

A natural question is whether it is possible to decide these equivalences for larger classes of nets, notably unbounded P/T nets. However, as Esparza observed in [7], all the behavioral equivalences ranging from interleaving bisimilarity to fully-concurrent bisimilarity are undecidable on unbounded P/T nets. So, there is no hope to extend our result about fc-bisimilarity further. Nonetheless, the proof of undecidability by Jančar [13] does not apply to causal-net bisimilarity, so that the decidability of causal-net bisimilarity over unbounded P/T nets is open. As a future work, we plan to extend Vogler's results in [20] about decidability of weak fully-concurrent bisimilarity on safe nets with silent moves, to bounded nets with silent moves, by means of our indexed marking idea.

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