Two Papers of Henri Poincaré on Mathematical Physics

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1921

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by H. A. Lorentz

The following pages cannot at all give a complete idea of what theoretical physics owes to Poincaré. I would have been happy to pay homage to his memory by presenting to the reader such a general picture, but I moved back in front of this task, that cannot be done with dignity without long and serious studies for which there was no time for me. I limited thus myself to two papers, that on the Dynamics of the electron, written in 1905 and published the following year in *Rendiconti del Circolo Matematico di Palermo*, and the study on the quantum theory which appeared in the *Journal de Physique* at the beginning of 1912.

To fully appreciate the first of this works, I will have to enter in some details on the ideas whose development led to the principle of relativity. Thus let us speak a little about the part that I contributed to this development, I must say first that I have found a valuable encouragement in the benevolent interest that Poincaré constantly took with my studies. Moreover, we will see soon by which degree he surpassed me.

It is known that Fresnel had based the explanation of the astronomical aberration on the assumption of a motionless ether that the celestial bodies would cross without entraining it. We also know his famous theorem, a necessary complement of this fundamental assumption, of the partial entrainment of light waves by moving matter. An transparent body animated by translation will communicate to the rays only a fraction of its own speed, a fraction which is determined by the "coefficient of Fresnel" $1\frac{1}{n^2}$, in which N is the index of refraction of the medium.

When, thanks to the work of Clerk Maxwell, our views on the nature of light had been profoundly changed, it was natural to try a deduction of this coefficient based on the principles of the electromagnetic theory. That's the goal I set myself, which could be achieved without too much difficulty in the theory of electrons.

The majority of the phenomena which are connected to aberration, and in particular the absence of an influence of the earth's motion in all the experiments where the collective system of devices is at rest in respect to our planet, could now be explained in a satisfactory way. It was only necessary to make the restriction, that the considered effects were of first order of magnitude compared to the speed of the Earth divided by the speed of light, terms of the second order have been neglected in calculations.

However, in 1881 Mr. Michelson succeeded to interfere two light rays, that were departed from a single point and came back after following rectilinear and mutually perpendicular paths of equal length. He found that the observed phenomena are again insensitive to the earth's motion; the interference fringes preserved the same positions, whatever were the directions of the arms of the device.

This time it was indeed an effect of the second order and it was easy to see that the hypothesis of the stationary aether alone is not sufficient for the explanation of the negative result. I was obliged to make a new assumption which consists in admitting, that the translation of a body through the aether produces a slight contraction of the body in the direction of motion. This assumption was indeed the only possibility; it had also been imagined by Fitzgerald and it found the approval of Poincaré, who however did not conceal the lack of satisfaction that the theories gave him in which one multiplies special assumptions invented for particular phenomena. This criticism was for me an additional reason to seek a general theory, in which the same principles would lead to the explanation of the experiment of Mr. Michelson and all those that were undertaken after him to discover effects of the second order. In the theory that I proposed, the absence of phenomena due to the collective motion of a system should be demonstrated for any value of speed less than that of light.

The method to be followed was indeed indicated. It was obviously necessary to demonstrate that the phenomena which take place in a material system can be represented by equations of the same form, the system may be at rest or being animated by a uniform translatory motion, and this equality of form has to be obtained using a suitable substitution of new variables. It was a question of finding transformation formulas, suitable for the independent variables, the coordinates *x*, *y*, *z* and time *t*, as well as for the various physical magnitudes, speeds, forces, etc, and by showing the invariance of the equations for these transformations.

The formulas that I established for coordinates and time can be put as[1]

(1) $x' = kl(x + \epsilon t),\ y' = ly,\ z' = lz,\ t' = kl(t + \epsilon x),$

where *ε*, *k*, *l* are constants which are, however, reduced to one. We see immediately that the origin of the new coordinates ($x' = 0$) is

$$x = -\epsilon t$$

so the point moves in the system *x*, *y*, *z*, *t* with speed -*ε* in the direction of the *x*-axis. The coefficient *k* is defined by

$$k = \left(1 - \epsilon^2\right)^{-\frac{1}{2}}$$

and *ε* is a function of *l* that has the value 1 for *ε* = 0. I initially left it undetermined, but I found in the course of my calculations, that to obtain the invariance (that I had in mind) we must put *l* = 1.

These were the considerations published by me in 1904 which gave place to Poincaré to write his paper on the dynamics of the electron, in which he attached my name to the transformation to which I will come to speak. I must notice on this subject that the same transformation was already present in an article of Mr. Voigt published in 1887, and that I did not draw from this artifice all the possible parts. Indeed, for some of the physical quantities which enter the formulas, I did not indicate the transformation which suits best. That was done by Poincaré and then by Mr. Einstein and Minkowski.

To find the "transformations of relativity", as I will call them now, it is sufficient in some cases to describe the phenomena in the system x', y', z', t' *exactly in the same way* as we do it in system *x, y, z, t*. Let us consider, for example, the motion of a point. If, in time *dt* the coordinates *x, y, z* undergo the changes *dx, dy, dz,* then we have for the velocity components

$$\xi = \frac{dx}{dt},\ \eta = \frac{dy}{dt},\ \zeta = \frac{dz}{dt}$$

However, by these relations the variations *dx, dy, dz, dt* contain the changes

(2) $dx' = kl(dx + \epsilon\, dt),\ dy' = l\, dy,\ dz' = l\, dz,\ dt' = kl(dt + \epsilon\, dx)$

of the new variables. It is natural to define the velocity components in the new system by the formulas

$$\xi' = \frac{dx'}{dt'},\ \eta' = \frac{dy'}{dt'},\ \zeta' = \frac{dz'}{dt'} \tag{3}$$

which gives us

$$(4) \qquad \xi' = \frac{\xi + \epsilon}{1 + \epsilon\xi},\ \eta' = \frac{\eta}{k(1 + \epsilon\xi)},\ \zeta' = \frac{\zeta}{k(1 + \epsilon\xi)}$$

To have another example, we can imagine a great number of mobile points whose velocities are continuous functions of the coordinates and time. Let *dτ* an element of volume located at point *x, y, z* and let us fix the attention to the points of the system which are in this element at one given moment *t*. Let t'_0 be the special value of t' which corresponds to *x, y, z, t* by the equations (1), and consider for the various points the values of x', y', z' which correspond to the given value $t' = t'_0$; in other words, let us consider the positions of the points in the new system, all taken for the same value of "time" t'. One might ask after the extension of the element $d\tau'$ of space x', y', z', in which are at this moment t'_0 the selected points which are in *dτ* at the time *t*. A simple calculation, which I omit here, led to the relation

$$(5) \qquad d\tau' = \frac{l^3}{k}\frac{1}{1 + \epsilon\xi}d\tau$$

Finally, let us suppose that the points in question carry equal electric charges, and admit that in two systems *x, y, z, t* and x', y', z', t' we attribute the same numerical values to these charges. If the points are sufficiently close to each other, we obtain a continuous distribution of electricity and it is clear that the charge contained in the element *dτ* at the moment *t* is equal to that which is in $d\tau'$ at the moment t'. Consequently, if *ρ* and ρ' are the densities of these charges,

$$\rho\, d\tau = \rho' d\tau' \tag{6}$$

and, by virtue of (5)

$$\rho' = \frac{k}{l^3}(1+\epsilon\xi)\rho \tag{7}$$

By this formula, combined with (4), we deduce again

$$\rho'\xi' = \frac{k}{l^3}\rho(\xi+\epsilon),\ \rho'\eta' = \frac{1}{l^3}\rho\eta,\ \rho'\zeta' = \frac{1}{l^3}\rho\zeta$$

These are the transformation formulas for the convection current.

For other physical quantities such as electric and magnetic forces, it is necessary to follow a less direct method; we will seek, perhaps with a little groping, the formulas of transformation suitable to ensure the invariance of the electromagnetic equations.

The formulas (4) and (7) are not in my memoir of 1904. Because I had not thought of the direct way which led there, and because I had the idea that there is an essential difference between systems *x, y, z,* t and x', y', z', t'. In one we use - such was my thought - coordinate axes which have a fixed position in the aether and which we can call "true" time; in the other system, on the contrary, we would deal with simple auxiliary quantities whose introduction is only a mathematical artifice. In particular, the variable t' could not be called "time" in the same way as the variable *t*.

In this order of ideas I did not think of describing the phenomena in the system x', y', z', t', exactly in the same way as in system *x*, *y*, *z*, *t*, and I did not define by the equations (3) and (7) the quantities ξ', η', ζ', ρ' which will correspond to ξ, η, ζ, ρ. It is rather by groping that I arrived at my formulas of transformation which, with our current notation, take the form

$$\xi' = k^2(\xi + \epsilon),\ \eta' + k\eta,\ \zeta' = k\zeta,\ \rho' = \frac{1}{kl^3}\rho$$

and that I wanted to choose, so as to obtain in the new system the simplest equations. Later, I could see in the paper of Poincaré that when proceeding more systematically I could have reached an even greater simplification. Not having noticed it, I did not succeed in obtaining the exact invariance of the equations; my formulas remained encumbered with certain terms which should have disappeared. These terms were too small to have an appreciable effect on the phenomena and I could thus explain the independence of the earth's motion that was revealed by observations, but I did not establish the principle of relativity as rigorously and universally true.

Poincaré, on the contrary, obtained a perfect invariance of the equations of electrodynamics, and he formulated the "postulate of relativity", terms which he was the first to employ. Indeed, stating from the point of view that I had missed, he found the formulas (4) and (7). Let us add that by correcting the imperfections of my work he never reproached me for them.

I can not explain here all the beautiful results obtained by Poincaré. Let us insist however on some points. Initially, he was not satisfied to show that the transformations of relativity leave intact the form of the electromagnetic equations. He explains the success of substitutions by noticing that these equations can be put in the form of the principle of least action and that the fundamental equation which expresses this principle, as well as the operations by which we deduce the field equations, are the same in systems *x, y, z, t* and x', y', z', t'.

In the second place, in accordance with the title of his paper, Poincaré particularly considers the way in which the deformation of a moving electron occurs, comparable with that of the arms of the device of Mr. Michelson, which is required by the postulate of relativity. Two different assumptions had been proposed on this subject. According to both an electron, presumably spherical in the state of rest, would change by a translation into an oblated ellipsoid of revolution, the axis of symmetry coincide with the direction of motion and the ratio of this axis to the diameter of the equator being given by $\sqrt{1-v^2}$,, if *v* is the velocity. But the assumptions differed between them with regard to the length of the axes and consequently the volume of the electron. While I had been led to admit that the radius of the equator remains equal to that of the original sphere, Mr. Bucherer and Mr. Langevin rather wanted to assign a constant size with volume. The first assumption corresponds to $l = 1$, the second with $kl^3 = 1$. Let us immediately add that the first value is the only one which is compatible with the postulate of relativity.

If one wants to realize the persistence and the balance of an electron while making use of the ordinary notions of mechanics, it is obviously not sufficient to consider the electrodynamic actions. The particle - that we consider here as a sphere carrying a surface charge - would immediately explode because of the mutual repulsions or, which is to same, of the stresses of Maxwell exerted on its surface. Therefore another concept should also be introduced, and Poincaré distinguishes at this place between the "bindings" and the "supplementary forces". He initially supposed that there is only the connection represented by the equation

$$r = b\theta^m$$

r is the semi-axis of the electron, $r\theta$ its equatorial radius, *b* and *m* variables that remain constant when *r* and θ (or one of these quantities) vary with the translation speed *v*. This granted, we know for any value of *v* the dimensions of the electron - because we know that $\theta = \left(1 - v^2\right)^{-\frac{1}{2}}$ - and by the ordinary formulas of electromagnetic field, the energy, momentum and the Lagrange function can be calculated. Between these values, considered as functions of *v*, there must be well known relations. Poincaré shows that they are verified only for $m = -\frac{2}{3}$, which brings us back to the constancy of volume, that is to say, the hypothesis of Mr. Bucherer and Langevin. But we know already that it is not this hypothesis, but only that of a constant equatorial radius, which is in agreement with the postulate of relativity. It is thus necessary to have recourse to additional forces.

By supposing that they depend on a potential of the form $Ar^{\alpha}\theta^{\beta}$, where A, α and β are constants, Poincaré finds that the constancy of the equatorial radius requires α = 3, β = 2, *i.e.* the potential in question must be proportional to volume. It results from it that the sought supplementary forces are equivalent to a pressure or a normal tension exerted on the surface and whose magnitude per unit of area remains constant, whatever the speed of translation. It is immediately seen that only a tension directed towards the interior is appropriate; we will determine the magnitude by the condition of an electron at rest and which has consequently the shape of a sphere, and it must be in equilibrium with the electrostatic repulsions. So when the particle is set into motion, the stress of Poincaré is united with the electrodynamic actions, and will inevitably produce the oblateness which is required by the principle of relativity.

After having found his supplementary force, Poincaré showed that the transformations of relativity do not change the form of the terms which it represents; thus he showed that *arbitrary* motions of a system of electrons can take place in the completely same manner in system *x, y, z, t* and in the system x', y', z', t'.

I already spoke about the necessity for posing *l* = 1 (constancy of the equatorial radius of the electron). I will not repeat here the demonstration given by Poincaré and I will only say that he showed the mathematical origin of this condition. One can consider all the transformations which are represented by formulas (1), with different values for speed -*ε*, and the corresponding values of *k* and *l*, this last coefficient has to be regarded as a function of *ε*; we can add to it other similar transformations which we deduce from (1) by changing the directions of the axes, and finally by arbitrary rotations. The postulate of relativity requires that all these transformations form a group and that is only possible if *l* has the constant value 1.

The "group of relativity" obtained, consists of linear substitutions which do not affect the quadratic form

$$x^2 + y^2 + z^2 - t^2$$

The paper ends with the application of the postulate of relativity on the phenomena of gravitation. Here, it is the question of finding the rule which determines the propagation of it, and the formulas which express the components of the force according to the coordinates and the speed, as well as of the attracted body as of the attracting body. By considering these questions, Poincaré starts by seeking the invariants of the group of relativity; indeed, it is clear that it must be possible to represent the phenomena by equations which contain only these invariants. However, the problem is undetermined. It is natural to admit that the propagation velocity is equal to that of light and that the variations of the law of Newton must be of second-order magnitude in respect to the velocities. But even with these restrictions, there is the choice between several assumptions, among which there are two that were especially indicated by Poincaré.

In this last part of the article one finds some new concepts which I must especially announce. Poincaré notices, for example, if *x*, *y*, *z* and $t\sqrt{-1}$ are considered as the coordinates of a point in four-dimensional space, the transformations of relativity are reduced to rotations in this space. He also had the idea of adding to the three force-components X, Y, Z the magnitude

$$T = X\xi + Y\eta + Z\zeta$$

which is nothing but the work of the force per unit time and which we can (to some extent) regard as a fourth component. When we ask after the force that a body experiences per unit volume, the magnitudes X, Y, Z, $T\sqrt{-1}$ are affected by a transformation of relativity in the same way as the magnitudes x, y, z, $t\sqrt{-1}$.

I recall these ideas of Poincaré because they are similar to methods that Minkowski and other scientists have later used to facilitate mathematical operations that arise in the theory of relativity.

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Let us pass now to the paper on the quantum theory. Towards the end of 1911 Poincaré had attended the meeting of the Council of Physics convened in Brussels by Mr. Solvay, in which we had especially dealt with the phenomena of the calorific radiation and the hypothesis of the elements or quanta of energy imagined by Mr. Planck to explain them. In the discussions, Poincaré had shown all the promptness and the penetration of his spirit and we had admired the facility with which he could enter the most difficult questions of Physics, even in those which were new for him. At the return to Paris, he did not cease dealing with the problem of which he felt the high importance. If the hypothesis of Mr. Planck were true, "the physical phenomena would cease obeying laws expressed by differential equations, and it would be, undoubtedly, the greatest and most profound revolution that natural philosophy suffered since Newton".

But are these new conceptions really inevitable and is there no way to arrive at the radiation law without introducing these discontinuities which are in direct opposition with the notions of traditional mechanics? Here is the question that Poincaré poses in his paper and to which he gives an answer that I will allow myself to briefly summarize.

Let us consider a system made up of *n* resonators of Planck and *p* molecules, *n* and *p* being very great numbers; let us suppose that all the resonators are equal between them and that it is the same for the molecules. Let us indicate by $\xi_1, \ldots, \xi_p$ the energies of the molecules and by $\eta_1, \ldots, \eta_n$ those of the resonators; each one of these variables will be able to take all the positive values.

Poincaré showed first that the probability so that the quantities of energy are between the limits ξ_1 and $\xi_1 + d\xi_1, \ldots, \xi_p$, and $\xi_p + d\xi_p$, η_1 and $\eta_1 + d\eta_1, \ldots, \eta_n$, η_n and $\eta_n + d\eta_n$, can be represented by

$$\omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) d\eta_1 \ldots d\eta_n d\xi_1 \ldots d\xi_p$$

where ω is a function for which we can make different hypotheses.

Once we know this function we can tell how much energy *h* will be distributed over the molecules and resonators. For this purpose, we can imagine a space of *p* + *n* dimensions, $\xi_1, \ldots, \xi_\rho, \eta_1, \ldots, \eta_n$, the infinitely thin layer S, in which the total energy

$$\xi_1 + \cdots + \xi_p + \eta_1 + \cdots + \eta_n$$

lies between *h* and an infinitely close value *h* + *dh*. The three integrals will be calculated

$$I = \int \omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) d\eta_1 \ldots d\eta_n d\xi_1 \ldots d\xi_p$$

$$I' = \int x\omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) d\eta_1 \ldots d\eta_n d\xi_1 \ldots d\xi_p$$

$$I'' = \int (h-x)\omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) d\eta_1 \ldots d\eta_n d\xi_1 \ldots d\xi_p$$
$$\left(x = \eta_1 + \cdots + \eta_n\right)$$

extended to the layer S, and we have $\frac{I'}{I}$ for the energy that the resonators take, and $\frac{I''}{I}$ for that of all the molecules. Therefore, if Y is the mean energy of a resonator, and X is that of a molecule,

$$nYI = I',\ pXI = I''$$

To calculate the integral I, we may first give fixed values to variables $\eta_1, \ldots, \eta_n$ and consequently to their sum *x*, and extend the integration over ξ for all positive values of these variables, for which the sum $\xi_1 + \cdots + \xi_p$ is between *h* - *x* and *h* - *x* + *dh*. This gives us

$$\int d\xi_1 \ldots d\xi_p = \frac{1}{(p-1)!}(h-x)^{p-1} dh$$

Then we can calculate the integral

$$\int \omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) d\eta_1 \ldots d\eta_n$$

extended to positive values of η such that $\eta_1 + \cdots + \eta_p$ lies between *x* and *x* + *dx*. Let

(8) $$\int \omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) d\eta_1 \ldots d\eta_n = \varphi(x) dx$$

φ is a function that depends on the function ω and we have

$$I = \frac{dh}{(p-1)!} \int_0^h (h-x)^{p-1} \varphi(x) dx$$

I' and I'' are calculated in the same manner, we only need to introduce under the sign of integration the factor *x* or the factor *h* - *x*. Ultimately, we can write

(9) $$nY = C \int_0^h x(h-x)^{p-1} \varphi(x) dx$$

(10) $$pX = C \int_0^h (h-x)^p \varphi(x) dx$$

where the factor C is the same in both cases. We do not have to deal with it because it is sufficient to determine the ratio of X to Y.

Now we obtain the Planck formula - which can be regarded as an expression of reality - if we make on the function ω the following hypothesis, which is consistent with quantum theory.

Let ε be the magnitude of the quantum of energy which is specific to the resonators considered, and denote by δ an infinitely small quantity[2]. The function ω is zero, except in the intervals

$$k\epsilon < \eta < k\epsilon + \delta$$

and for each of these intervals the integral $\int_{k\epsilon}^{k\epsilon+\delta} \omega \, d\eta$ has the value 1.

These data are sufficient for determining the function φ and the ratio $\frac{Y}{X}$ for which we find, as I said before, the value given by Planck's theory. I did not stop at these calculations and I pass immediately to the principal question, whether the discontinuities that I just mentioned must necessarily be admitted.

I will reproduce the reasoning of Poincaré, but I will at first say that in the formulas that we will encounter, α indicates a complex variable of which the real part α_r is always positive. In the representation we will limit ourselves to the half of plane α characterized by $\alpha_r > 0$, and in integrations in respect to α we will follow a straight line *l* perpendicular to the axis of real α, and prolonged indefinitely on the two sides. The values of the integrals will be independent of the length of the distance $\alpha_r > 0$ of this line at the origin of α.

Poincaré introduced an auxiliary function that defines the equation

$$\Phi(\alpha) = \int_0^\infty \omega(\eta) e^{-\alpha\eta} d\eta \tag{11}$$

and demonstrated that the function ω and the derived function φ can be be expressed by using Φ.

We obtain at first, by inverting (11)

$$\omega(\eta) = \frac{1}{2i\pi} \int_{(L)} \Phi(\alpha) e^{\alpha\eta} d\alpha \tag{12}$$

For a similar formula for $\varphi(x)$ we notice that in equation (11) we can replace η by any of the variables $\eta_1, \ldots, \eta_n$. Multiplying the *n* equations which we obtained, we find

$$[\Phi(\alpha)]^n = \int_0^\infty \ldots \int_0^\infty \omega\left(\eta_1\right) \ldots \omega\left(\eta_n\right) e^{-\alpha x} d\eta_1 \ldots d\eta_n$$

or, by virtue of the formula (8)

$$[\Phi(\alpha)]^n = \int_0^\infty \varphi(x)^{-\alpha x} dx$$

and by inversion

$$\varphi(x) = \frac{1}{2\pi i} \int_{(L)} [\Phi(\alpha)]^n e^{\alpha x} d\alpha$$

The formulas (9) and (10) now become

$$nY = \frac{C}{2i\pi} \int_0^h \int_{(L)} x(h-x)^{p-1} [\Phi(\alpha)]^n e^{\alpha x} dx\, d\alpha$$

$$pX = \frac{C}{2i\pi} \int_0^h \int_{(L)} (h-x)^p [\Phi(\alpha)]^n e^{\alpha x} dx\, d\alpha$$

and Poincaré again transforms them by substitutions

$$x = n\omega,\ h = n\beta,\ p = nk$$

which give

$$nY = \frac{Cn^{p+1}}{2i\pi} \int_0^\beta \int_{(L)} \frac{\omega}{\beta-\omega} \Theta^n d\omega\, d\alpha$$

$$pX = \frac{Cn^{p+1}}{2i\pi} \int_0^\beta \int_{(L)} \Theta^n d\omega\, d\alpha$$

he posed

$$\Theta = \Phi(\alpha) e^{\alpha\omega} (\beta-\omega)^k$$

Note that ω is nothing else than the average energy of a single resonator for the case

$$\eta_1 + \cdots + \eta_n = x$$

that β is the value that would be ω if all available energy *h* was in the resonator, and that *k* is the ratio between the number of molecules and the resonators.

When, in the applications of the probability theory to the molecular theories, we seek the state of a system that presents the maximum of probability, we always find that, thanks to the immense number of the molecules, this maximum is so pronounced that one can neglect the probability of all the states which deviate appreciably from the most probable state. In the case which occupies us, there is something similar.

Let us admit with Poincaré that, for values given of *h* and β, the function Θ has a maximum for $\alpha = \alpha_0$, $\omega = \omega_0$ and passes through the point α_0, the place of the maximum, the line *l* whose distance α_0 in the beginning could be selected at will. As the exponent *n* is very high, the maximum of Θ^n is extremely pronounced and the only elements of the integrals which we have to take into account, are those who are in the immediate vicinity of α_0 and of ω_0. That immediately gives us for the sought ratio

$$\frac{nY}{pX} = \frac{\omega_0}{\beta - \omega_0}$$

and, by virtue of the equation

$$nY = pX = h = n\beta$$

(13) $$Y = \omega 0$$

(14) $$X = \frac{\beta - \omega_0}{k}$$

To determine the values of α_0 and ω_0, we can use equations

$$\frac{\partial \log \Theta}{\partial \alpha} = 0, \ \frac{\partial \log \Theta}{\partial \omega} = 0$$

from which we derive

(15) $$\frac{\Phi'(\alpha_0)}{\Phi(\alpha_0)} + \omega_0 = 0$$

and

(16) $$\alpha_0 - \frac{k}{\beta - \omega_0} = 0$$

We see from these formulas that α_0 and ω_0 depend on the quantity β, that is to say the total amount of energy *h* which was communicated to the system; this is a result which was to be expected. Equation (16) tells us further that α_0 will always be real. This quantity determines immediately the average energy of a molecule as it follows from (14) and (16)

$$X = \frac{I}{\alpha_0}$$

Now we see that the average energy of a molecule is proportional to absolute temperature T. We can write

$$\alpha_0 = \frac{c}{T}$$

where *c* is a known constant, and equation

$$Y = -\frac{\Phi'(\alpha_0)}{\Phi(\alpha_0)} \tag{17}$$

which we draw from (13) and (15), gives us the average energy as a function of temperature. We see that this result is independent of the ratio between the numbers *n* and *p*.

Suppose now that we know for all temperatures the average energy of a resonator. By (17) we will thus know for all positive values of α the derivative $\frac{d \log \Phi(\alpha)}{d\alpha}$; we will deduce from them Φ(α) except for a constant factor. Of course, these findings will at first be limited to real values of α, but the function Φ(α) is assumed to be as determined throughout the semi-plane α about which we spoke, when it is given at all points of the real and positive semi-axis.

Finally, the formula (12) will provide us the function of probability ω for an unspecified positive value of η. It is true that the unspecified factor of function Φ(α) will be found in ω, but such a factor does not have any importance.

We can thus say that the probability ω is entirely given as soon as we know the distribution of energy *for all temperatures*. There is only *one* function ω for a distributions which is given as a function of the temperature. Consequently, the assumptions that we made on ω and which lead to the law of Planck are the only ones that we can admit.

That is the reasoning by which Poincaré established the *necessity* of the quantum hypothesis.

We see that the conclusion depends on the assumption that Planck's formula is an accurate image of reality. This could be drawn into question, and the formula could only be approximate. It is for this reason that Poincaré takes up the problem by abandoning Planck's law and using only the relationship that this physicist has found between the energy of a resonator and that of black body radiation. The reassessment led to the conclusion that the total energy of the radiation will be infinite unless the integral $\int_0^{\eta_0} \omega \, d\eta$ does not tend to zero with η_0. The function ω must have at least *one* discontinuity (for η = 0), similar to those given by quantum theory[3].

1. ↑ I follow here the notations of Poincaré and I choose the units of length and time so that the speed of light is equal to 1.
2. ↑ This is the *first* theory of Planck, in which it is assumed that the energy of a resonator can only have values 0, ε, 2ε, 3ε, etc..
3. ↑ This result was found by P. Ehrenfest, see *Ann. Physik*, t. 36, 1911, p. 91.

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- Dbenbenn
- Jacobolus
- Indolences
- Technion
- Dha
- Abigor
- Reisio
- Blurpeace
- Dschwen
- Steinsplitter

- KABALINI
- Bromskloss
- Tene~commonswiki
- AzaToth
- Bender235
- PatríciaR